

Tridiagonal canonical matrices of bilinear or sesquilinear forms and of pairs of symmetric, skew-symmetric, or Hermitian forms*

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Abstract

Tridiagonal canonical forms of square matrices under congruence or *congruence, pairs of symmetric or skew-symmetric matrices under congruence, and pairs of Hermitian matrices under *congruence are given over an algebraically closed field of characteristic different from 2.

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1 Introduction

We give tridiagonal canonical forms of matrices of

- (i) bilinear forms and sesquilinear forms,
- (ii) pairs of forms, in which each form is either symmetric or skew-symmetric, and
- (iii) pairs of Hermitian forms

over an algebraically closed field of characteristic different from 2. Our canonical forms are direct sums of matrices or pairs of matrices of the form

$$\begin{bmatrix} \varepsilon & a & & & 0 \\ a' & 0 & b & & \\ & b' & 0 & a & \\ & & a' & 0 & b \\ & & & b' & 0 & \ddots \\ 0 & & & & \ddots & \ddots \end{bmatrix}; \quad (1)$$

they employ relatively few different types of canonical direct summands.

Let \mathbb{F} be a field of characteristic different from 2. The problem of classifying bilinear or sesquilinear forms over \mathbb{F} was reduced by Gabriel, Riehm, and Shrader-Frechette [5, 17, 18] to the problem of classifying Hermitian forms over finite extensions of \mathbb{F} . In [22] this reduction was extended to selfadjoint representations of linear categories with involution, and canonical matrices of (i)–(iii) were obtained over \mathbb{F} up to classification of Hermitian forms over finite extensions of \mathbb{F} . Canonical matrices were found in a simpler form in [9] when $\mathbb{F} = \mathbb{C}$. Canonical matrices of bilinear forms over an algebraically closed field of characteristic 2 were given in [21]. The problem of classifying pairs of symmetric, skew-symmetric, or Hermitian forms was studied by many authors; we refer the reader to Thompson’s classical work [24] with a bibliography of 225 items, and to the recent papers by Lancaster and Rodman [11, 12].

Each $n \times n$ matrix A over \mathbb{F} defines a bilinear form $x^T A y$ on \mathbb{F}^n . If \mathbb{F} is a field with a fixed nonidentity involution $a \mapsto \bar{a}$, then A defines a sesquilinear form $\bar{x}^T A y$ on \mathbb{F}^n . Two square matrices A and A' give the same bilinear (sesquilinear) form with respect to different bases if and only if they

are *congruent* (**congruent*); this means that there is a nonsingular S such that $S^T A S = A'$ ($S^* A S = A'$ with $S^* := \bar{S}^T$, respectively). Two matrix pairs (A, B) and (A', B') are *congruent* (**congruent*) if there is a nonsingular S such that $S^T A S = A'$ and $S^T B S = B'$ ($S^* A S = A'$ and $S^* B S = B'$, respectively). A matrix A is *Hermitian* if $A = A^*$.

Thus, the canonical form problem for (i)–(iii) is the canonical form problem for

- (i') matrices under congruence or *congruence (their tridiagonal canonical matrices are given in Theorems 1.1 and 1.2);
- (ii') pairs of matrices under congruence, in which each matrix is either symmetric or skew-symmetric (Theorems 3.1–5.1); and
- (iii') pairs of Hermitian matrices under *congruence (Theorem 8.1).

The problem of finding tridiagonal canonical forms of (ii') or (iii') is connected with the problem of tridiagonalizing matrices by orthogonal or unitary similarity: two pairs (I_n, B) and (I_n, B') are congruent or *congruent if and only if B and B' are orthogonally or unitarily similar, respectively. The well-known algorithm for reducing symmetric real matrices to tridiagonal form by orthogonal similarity [26, Section 5] can not be extended to symmetric complex matrices. However, Ikramov [10] showed that every symmetric complex matrix is orthogonally similar to a tridiagonal matrix. Each 4×4 complex matrix is unitarily similar to a tridiagonal matrix [1, 16], but there is a 5×5 matrix that is not unitarily similar to a tridiagonal matrix [1, 4, 14, 23].

Our paper was inspired by [3], in which Đoković and Zhao gave a tridiagonal canonical form of symmetric matrices for orthogonal similarity over an algebraically closed field of characteristic different from 2 (we use it in Theorem 3.2 of our paper). In a subsequent article, and for the same type of field, Đoković, Rietsch, and Zhao [2] found a 4-diagonal canonical form of skew-symmetric matrices for orthogonal similarity.

Matrix pairs (A, B) and (A', B') are *equivalent* if there are nonsingular R and S such that $R A S = A'$ and $R B S = B'$. We denote equivalence of pairs by \approx . Kronecker's theorem on pencils of matrices [6, Section XII, Theorem 5] ensures that each pair of matrices of the same size is equivalent to a direct sum, determined uniquely up to permutation of summands, of pairs of the form

$$(I_n, J_n(\lambda)), \quad (J_n(0), I_n), \quad (F_n, G_n), \quad (F_n^T, G_n^T),$$

in which I_n is the $n \times n$ identity matrix,

$$J_n(\lambda) := \begin{bmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix} \text{ is } n\text{-by-}n,$$

and

$$F_n := \begin{bmatrix} 1 & 0 & & 0 \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{bmatrix} \text{ and } G_n := \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \end{bmatrix} \text{ are } n\text{-by-}(n+1).$$

Thus, $F_0 = G_0$ is the 0-by-1 matrix, which represents the linear mapping $\mathbb{F} \rightarrow 0$.

In the following two theorems (proved in Sections 6 and 7) we give tridiagonal canonical forms of a square matrix A under congruence and *congruence. We also give the Kronecker canonical form of (B^T, B) and, respectively, (B^*, B) for each canonical direct summand B , which permits us to construct the canonical form of A for congruence using the Kronecker canonical form of (A^T, A) , and to construct, up to signs of the direct summands, the canonical form of A for *congruence using the Kronecker canonical form of (A^*, A) .

Theorem 1.1. (a) *Each square matrix A over an algebraically closed field \mathbb{F} of characteristic different from 2 is congruent to a direct sum, determined uniquely up to permutation of summands, of tridiagonal matrices of three types:*

$$\begin{bmatrix} 0 & 1 & & 0 \\ \lambda & 0 & 1 & \\ & \lambda & 0 & \ddots \\ & & \ddots & \ddots & 1 \\ 0 & & & \lambda & 0 \end{bmatrix}_{2k}, \quad \lambda \in \mathbb{F}, \lambda \neq \pm 1, \quad (2)$$

in which each nonzero λ is determined up to replacement by λ^{-1} (i.e., the matrices (2) with λ and λ^{-1} are congruent);

$$\begin{bmatrix} \varepsilon & 1 & & & 0 \\ -1 & 0 & 1 & & \\ & 1 & 0 & 1 & \\ & & -1 & 0 & 1 \\ & & & 1 & 0 & \ddots \\ 0 & & & & \ddots & \ddots \end{bmatrix}_n, \quad \begin{array}{l} \varepsilon = 1 \text{ if } n \text{ is a multiple of } 4, \\ \varepsilon \in \{0, 1\} \text{ otherwise;} \end{array} \quad (3)$$

and

$$\begin{bmatrix} 0 & 1 & & & 0 \\ 1 & 0 & 1 & & \\ & -1 & 0 & 1 & \\ & & 1 & 0 & 1 \\ & & & -1 & 0 & \ddots \\ 0 & & & & \ddots & \ddots \end{bmatrix}_{4k}. \quad (4)$$

The subscripts $2k$, n , and $4k$ (with $k, n \in \mathbb{N}$) designate the sizes of the corresponding matrices.

(b) The direct sum asserted in (a) is determined uniquely up to permutation of summands by the Kronecker canonical form of (A^T, A) for equivalence. For each direct summand B of types (2)–(4), the Kronecker canonical form of (B^T, B) is given in the following table:

B	Kronecker canonical form of (B^T, B)	
Matrix (2)	$(I_k, J_k(\lambda)) \oplus (J_k(\lambda), I_k)$	
Matrix (3)	$(F_k, G_k) \oplus (F_k^T, G_k^T)$ if $n = 2k + 1$	
with $\varepsilon = 0$	$(I_k, J_k(-1)) \oplus (I_k, J_k(-1))$ if $n = 2k$ (k is odd)	(5)
Matrix (3)	$(I_n, J_n((-1)^{n+1}))$	
with $\varepsilon = 1$		
Matrix (4)	$(I_{2k}, J_{2k}(1)) \oplus (I_{2k}, J_{2k}(1))$	

Let \mathbb{F} be an algebraically closed field with nonidentity involution. Fix $i \in \mathbb{F}$ such that $i^2 = -1$. It is known (see Lemma 2.1) that each element of \mathbb{F} is uniquely representable in the form $a + bi$ with a, b in $\mathbb{P} := \{\lambda \in \mathbb{F} \mid \bar{\lambda} = \lambda\}$, and the involution on \mathbb{F} is “complex conjugation”: $\overline{a + bi} = a - bi$. Moreover,

\mathbb{P} is ordered and $a^2 + b^2$ has a unique positive real root, which is called the *modulus* of $a + bi$ and is denoted by $|a + bi|$.

Theorem 1.2. (a) *Each square matrix A over an algebraically closed field \mathbb{F} with nonidentity involution is *-congruent to a direct sum, determined uniquely up to permutation of summands, of tridiagonal matrices of two types:*

$$\begin{bmatrix} 0 & 1 & & 0 \\ \lambda & 0 & 1 & \\ & \lambda & 0 & \ddots \\ & & \ddots & \ddots & 1 \\ 0 & & & \lambda & 0 \end{bmatrix}_n, \quad \begin{array}{l} \lambda \in \mathbb{F}, \quad |\lambda| \neq 1, \\ \text{each nonzero } \lambda \text{ is determined} \\ \text{up to replacement by } \bar{\lambda}^{-1}, \\ \lambda = 0 \text{ if } n \text{ is odd} \end{array} \quad (6)$$

(one can take $|\lambda| < 1$ if n is even); and

$$\mu \begin{bmatrix} 1 & 1 & & 0 \\ -1 & 0 & 1 & \\ & 1 & 0 & 1 \\ & & -1 & 0 & 1 \\ & & & 1 & 0 & \ddots \\ 0 & & & & \ddots & \ddots \end{bmatrix}_n, \quad \mu \in \mathbb{F}, \quad |\mu| = 1. \quad (7)$$

(b) *The Kronecker canonical form of (A^*, A) under equivalence determines the direct sum asserted in (a) uniquely up to permutation of summands and multiplication of any direct summand of type (7) by -1 . For each direct summand B of type (6) or (7), the Kronecker canonical form of (B^*, B) is given in the following table:*

B	Kronecker canonical form of (B^*, B)
Matrix (6)	$(F_k, G_k) \oplus (F_k^T, G_k^T) \quad \text{if } n = 2k + 1$ $(J_k(\bar{\lambda}), I_k) \oplus (I_k, J_k(\lambda)) \quad \text{if } n = 2k$
Matrix (7)	$(I_n, J_n((-1)^{n+1} \bar{\mu}^{-1} \mu))$

(8)

2 Four lemmas

In this section we prove four lemmas that we use in later sections. In the first lemma we collect known results about algebraically closed fields with involution; i.e., a bijection $a \mapsto \bar{a}$ satisfying $\overline{a + b} = \bar{a} + \bar{b}$, $\overline{ab} = \bar{a}\bar{b}$ and $\bar{\bar{a}} = a$.

Lemma 2.1. *Let \mathbb{F} be an algebraically closed field with nonidentity involution $\lambda \mapsto \bar{\lambda}$, and let*

$$\mathbb{P} := \{\lambda \in \mathbb{F} \mid \bar{\lambda} = \lambda\}. \quad (9)$$

Then \mathbb{F} has characteristic 0,

$$\mathbb{F} = \mathbb{P} + \mathbb{P}i, \quad i^2 = -1, \quad (10)$$

and the involution has the form

$$\overline{a + bi} = a - bi, \quad a, b \in \mathbb{P}. \quad (11)$$

Moreover, the field \mathbb{P} has a unique linear ordering \leq such that

$$a > 0 \text{ and } b > 0 \implies a + b > 0 \text{ and } ab > 0.$$

The positive elements of \mathbb{P} with respect to this ordering are the squares of nonzero elements. Every algebraically closed field of characteristic 0 possesses a nonidentity involution.

Proof. If \mathbb{F} is an algebraically closed field with nonidentity involution $\lambda \mapsto \bar{\lambda}$, then this involution is an automorphism of order 2. Hence \mathbb{F} has degree 2 over the field \mathbb{P} defined in (9). By Corollary 2 in [13, Chapter VIII, §9], \mathbb{P} has characteristic 0 and every element of \mathbb{F} is uniquely representable in the form $a + bi$ with $a, b \in \mathbb{P}$. Since the involution is an automorphism of \mathbb{F} , $\bar{i}^2 = -1$. So $\bar{i} = -i$ and the involution is (11). Due to Proposition 3 in [13, Chapter XI, §2], \mathbb{P} is a real closed field, and so the statements about the ordering \leq follow from [13, Chapter XI, §2, Theorem 1]. By [25, §82, Theorem 7c], every algebraically closed field of characteristic 0 contains at least one real closed subfield and hence it can be represented in the form (10) and possesses the involution (11). \square

The canonical form problem for pairs of symmetric or skew-symmetric matrices under congruence reduces to the canonical form problem for matrix pairs under equivalence due to the following lemma, which was proved in [15, §95, Theorem 3] for complex matrices. Roiter [19] (see also [20, 22]) extended this lemma to arbitrary systems of linear mappings and bilinear forms over an algebraically closed field of characteristic different from 2.

Lemma 2.2. *Let (A, B) and (A', B') be given pairs of $n \times n$ matrices over an algebraically closed field \mathbb{F} of characteristic different from 2. Suppose that*

A and A' are either both symmetric or both skew-symmetric, and also that B and B' are either both symmetric or both skew-symmetric. Then (A, B) and (A', B') are congruent if and only if they are equivalent.

Proof. If (A, B) and (A', B') are congruent then they are equivalent.

Conversely, let (A, B) and (A', B') be equivalent; i.e., $R^T AS = A'$ and $R^T BS = B'$ for some nonsingular R and S . Then

$$R^T AS = A' = \varepsilon(A')^T = \varepsilon S^T A^T R = S^T AR,$$

in which $\varepsilon = 1$ if A and A' are symmetric and $\varepsilon = -1$ if A and A' are skew-symmetric. Write $M := SR^{-1}$. Then

$$AM = M^T A, \quad AM^2 = (M^T)^2 A, \quad \dots$$

and so $Af(M) = f(M)^T A$ for every polynomial $f \in \mathbb{F}[x]$. If there exists $f \in \mathbb{F}[x]$ such that $f(M)^2 = M$, then for $N := f(M)R$ we have

$$N^T AN = R^T f(M)^T Af(M)R = R^T Af(M)^2 R = R^T AMR = R^T AS = A'.$$

Repeating the argument for the matrix B , we obtain $N^T BN = B'$. Consequently, (A, B) and (A', B') are congruent.

It remains to find $f \in \mathbb{F}[x]$ such that $f(M)^2 = M$. Let

$$(x - \lambda_1)^{k_1} \cdots (x - \lambda_t)^{k_t}, \quad \lambda_i \neq \lambda_j \text{ if } i \neq j,$$

be the characteristic polynomial of M . We can reduce M to Jordan canonical form and obtain

$$M = J_1 \oplus \cdots \oplus J_t, \quad J_i = \lambda_i I_{k_i} + F_i, \quad F_i^{k_i} = 0.$$

For the polynomial

$$\varphi_i(x) := \prod_{j \neq i} (x - \lambda_j)^{k_j}$$

we have

$$\varphi_i(M) = 0_{k_1 + \cdots + k_{i-1}} \oplus \varphi_i(J_i) \oplus 0_{k_{i+1} + \cdots + k_t} \quad (12)$$

(0_k denotes the $k \times k$ zero matrix). The field \mathbb{F} is algebraically closed of characteristic not 2, all λ_i and $\varphi_i(\lambda_i)$ are nonzero, so for each $i = 1, \dots, t$ there exist polynomials $\psi_i, \tau_i \in \mathbb{F}[x]$ such that

$$\psi_i(x)^2 \equiv \lambda_i + x, \quad \varphi_i(\lambda_i + x)\tau_i(x) \equiv \psi_i(x) \pmod{x^{k_i}}$$

(the coefficients of ψ_i and τ_i are determined successively from these congruences). Then $f(x) := \sum_i \varphi_i(x) \tau_i(x - \lambda_i)$ is the required polynomial. Indeed, by (12)

$$f(M) = \bigoplus_i \varphi_i(J_i) \tau_i(J_i - \lambda_i I_{k_i}) = \bigoplus_i \varphi_i(\lambda_i I_{k_i} + F_i) \tau_i(F_i) = \bigoplus_i \psi_i(F_i)$$

and so

$$f(M)^2 = \bigoplus_i \psi_i(F_i)^2 = \bigoplus_i (\lambda_i I_{k_i} + F_i) = \bigoplus_i J_i = M. \quad \square$$

For each matrix of the form

$$A = \begin{bmatrix} \varepsilon & a_1 & & & 0 \\ a'_1 & 0 & b_1 & & \\ & b'_1 & 0 & a_2 & \\ & & a'_2 & 0 & b_2 \\ & & & b'_2 & 0 & \ddots \\ 0 & & & & \ddots & \ddots \end{bmatrix}_n, \quad (13)$$

define

$$\mathcal{P}(A) := \begin{bmatrix} b_k & a'_k & & & & 0 \\ & \ddots & \ddots & & & \\ & & b_1 & a'_1 & & \\ & & & \varepsilon & a_1 & \\ & & & & b'_1 & \ddots \\ & & & & & \ddots & a_k \\ 0 & & & & & & b'_k \end{bmatrix}_{2k+1} \quad \text{if } n = 2k + 1$$

and

$$\mathcal{P}(A) := \begin{bmatrix} a_k & b'_{k-1} & & & & 0 \\ & a_{k-1} & \ddots & & & \\ & & \ddots & b'_1 & & \\ & & & a_1 & \varepsilon & \\ & & & & a'_1 & b_1 \\ & & & & & a'_2 & \ddots \\ & & & & & & \ddots & b_{k-1} \\ 0 & & & & & & & a'_k \end{bmatrix}_{2k} \quad \text{if } n = 2k.$$

Lemma 2.3. *Every pair (A, B) of $n \times n$ matrices of the form (13) is equivalent to $(\mathcal{P}(A), \mathcal{P}(B))$.*

Proof. If $n = 2k + 1$, then we rearrange rows $1, 2, \dots, 2k + 1$ in A and in B as follows:

$$2k, 2k - 2, \dots, 2, 1, 3, \dots, 2k - 1, 2k + 1,$$

and their columns in the inverse order:

$$2k + 1, 2k - 1, \dots, 3, 1, 2, \dots, 2k - 2, 2k.$$

If $n = 2k$, then we rearrange the rows of A and B as follows:

$$2k - 1, 2k - 3, \dots, 3, 1, 2, \dots, 2k - 2, 2k,$$

and their columns in the inverse order:

$$2k, 2k - 2, \dots, 2, 1, 3, \dots, 2k - 3, 2k - 1.$$

The pair that we obtain is $(\mathcal{P}(A), \mathcal{P}(B))$. □

For a sign $\sigma \in \{+, -\}$ and a nonnegative integer k , define the $2k$ -by- $2k$ matrix

$$M_k^\sigma := \begin{bmatrix} 0 & 1 \\ \sigma 1 & 0 \end{bmatrix} \oplus \dots \oplus \begin{bmatrix} 0 & 1 \\ \sigma 1 & 0 \end{bmatrix} \quad (k \text{ summands}).$$

Thus, M_0^σ is 0-by-0.

Lemma 2.4. *Let $\sigma, \tau \in \{+, -\}$ and $k \in \mathbb{N}$. Then the following pairs are equivalent:*

$$(0_1 \oplus M_k^\sigma, M_k^\tau \oplus 0_1) \approx (F_k, G_k) \oplus (F_k^T, G_k^T), \quad (14)$$

$$(I_1 \oplus M_k^\sigma, M_k^\tau \oplus 0_1) \approx (I_{2k+1}, J_{2k+1}(0)), \quad (15)$$

$$(0_1 \oplus M_{k-1}^\sigma \oplus 0_1, M_k^\tau) \approx (J_k(0), I_k) \oplus (J_k(0), I_k), \quad (16)$$

$$(I_1 \oplus M_{k-1}^\sigma \oplus 0_1, M_k^\tau) \approx (J_{2k}(0), I_{2k}). \quad (17)$$

Proof. Let $\varepsilon \in \{0, 1\}$. By Lemma 2.3,

$$([\varepsilon] \oplus M_k^\sigma, M_k^\tau \oplus 0_1) \approx (I_k \oplus [\varepsilon] \oplus I_k, J_{2k+1}(0)),$$

which proves (14) and (15), and

$$([\varepsilon] \oplus M_{k-1}^\sigma \oplus 0_1, M_k^\tau) \approx \left(\begin{bmatrix} 0 & 1 & & & 0 \\ & 0 & \cdot & & \\ & & \cdot & \varepsilon & \\ & & & \cdot & 1 \\ 0 & & & & 0 \end{bmatrix}, \begin{bmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & 1 \end{bmatrix} \right),$$

which proves (16) and (17). \square

3 Pairs of symmetric matrices

In this section, we give two tridiagonal canonical forms of pairs of symmetric matrices under congruence.

3.1 First canonical form

Theorem 3.1. (a) *Over an algebraically closed field \mathbb{F} of characteristic different from 2, every pair (A, B) of symmetric matrices of the same size is congruent to a direct sum, determined uniquely up to permutation of summands, of tridiagonal pairs of two types:*

$$\left(\begin{bmatrix} 0 & 1 & & & 0 \\ 1 & 0 & 0 & & \\ & 0 & 0 & 1 & \\ & & 1 & 0 & 0 \\ & & & 0 & 0 & \ddots \\ 0 & & & & \ddots & \ddots \end{bmatrix}_n, \begin{bmatrix} \varepsilon & \lambda & & & 0 \\ \lambda & 0 & 1 & & \\ & 1 & 0 & \lambda & \\ & & \lambda & 0 & 1 \\ & & & 1 & 0 & \ddots \\ 0 & & & & \ddots & \ddots \end{bmatrix}_n \right), \quad \begin{matrix} \lambda \in \mathbb{F}, \\ \varepsilon \in \{0, 1\}, \end{matrix} \quad (18)$$

in which $\varepsilon = 1$ if n is even and $\lambda = 0$ if n is odd; and

$$\left(\begin{bmatrix} 1 & 0 & & & 0 \\ 0 & 0 & 1 & & \\ & 1 & 0 & 0 & \\ & & 0 & 0 & 1 \\ & & & 1 & 0 & \ddots \\ 0 & & & & \ddots & \ddots \end{bmatrix}_n, \begin{bmatrix} \lambda & 1 & & & 0 \\ 1 & 0 & \lambda & & \\ & \lambda & 0 & 1 & \\ & & 1 & 0 & \lambda \\ & & & \lambda & 0 & \ddots \\ 0 & & & & \ddots & \ddots \end{bmatrix}_n \right), \quad (19)$$

in which $\lambda = 0$ if n is even and $\lambda \in \mathbb{F}$ if n is odd.

(b) This direct sum is determined uniquely up to permutation of summands by the Kronecker canonical form of (A, B) under equivalence. The Kronecker canonical form of each of the direct summands is given in the following table:

Pair	Kronecker canonical form of the pair	
(18)	$(F_k, G_k) \oplus (F_k^T, G_k^T)$ if $n = 2k + 1$ and $\varepsilon = 0$	(20)
	$(J_n(0), I_n)$ if n is odd and $\varepsilon = 1$	
	$(I_n, J_n(\lambda))$ if n is even	
(19)	$(I_n, J_n(\lambda))$ if n is odd	
	$(J_n(0), I_n)$ if n is even	

Proof. Let the Kronecker canonical form of (A, B) be

$$\bigoplus_i (I_{m_i}, J_{m_i}(\lambda_i)) \oplus \bigoplus_j (J_{n_j}(0), I_{n_j}) \oplus \bigoplus_l (F_{s_l}, G_{s_l}) \oplus \bigoplus_r (F_{t_r}^T, G_{t_r}^T).$$

Since A and B are symmetric,

$$(A, B) \approx \bigoplus_i (I_{m_i}, J_{m_i}(\lambda_i)) \oplus \bigoplus_j (J_{n_j}(0), I_{n_j}) \oplus \bigoplus_l (F_{s_l}^T, G_{s_l}^T) \oplus \bigoplus_r (F_{t_r}, G_{t_r}).$$

Thus, we can make $s_1 = t_1, s_2 = t_2, \dots$ by reindexing $\{t_r\}$, and obtain that the Kronecker canonical form of (A, B) is

$$\bigoplus_i (I_{m_i}, J_{m_i}(\lambda_i)) \oplus \bigoplus_j (J_{n_j}(0), I_{n_j}) \oplus \bigoplus_l \left((F_{s_l}, G_{s_l}) \oplus (F_{s_l}^T, G_{s_l}^T) \right). \quad (21)$$

This sum is determined by (A, B) uniquely up to permutation of summands. In view of Lemma 2.2, it remains to prove (20).

The pair (18) with $n = 2k + 1$ and $\varepsilon = 0$ has the form $(M_k^+ \oplus 0_1, 0_1 \oplus M_k^+)$; by (14) it is equivalent to $(F_k, G_k) \oplus (F_k^T, G_k^T)$.

The pair (18) with $n = 2k + 1$ and $\varepsilon = 1$ has the form $(M_k^+ \oplus 0_1, I_1 \oplus M_k^+)$; by (15) it is equivalent to $(J_{2k+1}(0), I_{2k+1})$.

The pair (18) with $n = 2k$ has the form $(M_k^+, \lambda M_k^+ + (I_1 \oplus M_{k-1}^+ \oplus 0_1))$; it is equivalent to $(I_{2k}, \lambda I_{2k} + J_{2k}(0)) = (I_{2k}, J_{2k}(\lambda))$ since (17) ensures that

$$(M_k^+, I_1 \oplus M_{k-1}^+ \oplus 0_1) \approx (I_{2k}, J_{2k}(0)). \quad (22)$$

The pair (19) with $n = 2k + 1$ has the form $(I_1 \oplus M_k^+, \lambda(I_1 \oplus M_k^+) + (M_k^+ \oplus 0_1))$; by (15) it is equivalent to $(I_{2k+1}, J_{2k+1}(\lambda))$.

The pair (19) with $n = 2k$ has the form $(I_1 \oplus M_{k-1}^+ \oplus 0_1, M_k^+)$; by (17) it is equivalent to $(J_{2k}(0), I_{2k})$. \square

3.2 Second canonical form

In this section, we give another tridiagonal canonical form of pairs of symmetric matrices for congruence. This form is not a direct sum of tridiagonal matrices of the form (1). It is based on the Đoković and Zhao's tridiagonal canonical form of symmetric matrices for orthogonal similarity [3] and resembles the Kronecker canonical form of matrix pairs for equivalence.

For each positive integer n , let N_n denote any fixed $n \times n$ tridiagonal symmetric matrix over \mathbb{F} that is similar to $J_n(0)$. Following [3, p. 79], we can take as N_n the value $N(a_1, \dots, a_n, b)$ of the polynomial matrix

$$N(x_1, \dots, x_n, y) := \begin{bmatrix} x_1 & y & & 0 \\ y & x_2 & \ddots & \\ & \ddots & \ddots & y \\ 0 & & y & x_n \end{bmatrix}$$

at any nonzero solution $(a_1, \dots, a_n, b) \in \mathbb{F}^{n+1}$ of the system

$$c_1(x_1, \dots, x_n, y) = 0, \dots, c_n(x_1, \dots, x_n, y) = 0$$

of equations whose left parts are the coefficients of the characteristic polynomial $t^n + c_1 t^{n-1} + \dots + c_n$ of $N(x_1, \dots, x_n, y)$. Then 0 is the only eigenvalue of N_n , $b \neq 0$, $\text{rank } N_n = n - 1$, and N_n is similar to $J_n(0)$.

If \mathbb{F} has the characteristic 0, then [3, p. 81] ensures that we can also take

$$N_n = \begin{bmatrix} n-1 & id_1 & & & 0 \\ id_1 & n-3 & id_2 & & \\ & id_2 & n-5 & \ddots & \\ & & \ddots & \ddots & id_{n-1} \\ 0 & & & id_{n-1} & 1-n \end{bmatrix}, \quad \begin{aligned} d_l &:= \sqrt{l(n-l)}, \\ i^2 &= -1. \end{aligned}$$

Theorem 3.2. *Over an algebraically closed field \mathbb{F} of characteristic different from 2, every pair (A, B) of symmetric matrices of the same size is congruent*

to a direct sum, determined uniquely up to permutation of summands, of tridiagonal pairs of three types:

$$(I_n, \lambda I_n + N_n) \text{ with } \lambda \in \mathbb{F}; \quad (N_n, I_n); \quad (23)$$

and

$$\left(\begin{bmatrix} 0 & 1 & & & 0 \\ 1 & 0 & 0 & & \\ & 0 & 0 & 1 & \\ & & 1 & 0 & 0 \\ & & & 0 & 0 & \ddots \\ 0 & & & & \ddots & \ddots \end{bmatrix}_{2k+1}, \begin{bmatrix} 0 & 0 & & & 0 \\ 0 & 0 & 1 & & \\ & 1 & 0 & 0 & \\ & & 0 & 0 & 1 \\ & & & 1 & 0 & \ddots \\ 0 & & & & \ddots & \ddots \end{bmatrix}_{2k+1} \right). \quad (24)$$

(b) This direct sum is determined uniquely up to permutation of summands by the Kronecker canonical form of (A, B) for equivalence. The Kronecker canonical form of each of the direct summands is given in the following table:

Pair	Kronecker canonical form of the pair
$(I_n, \lambda I_n + N_n)$	$(I_n, J_n(\lambda))$
(N_n, I_n)	$(J_n(0), I_n)$
(24)	$(F_k, G_k) \oplus (F_k^T, G_k^T)$

(25)

Proof. In view of (21) and Lemma 2.2, it suffices to prove (25). The equivalences

$$(I_n, \lambda I_n + N_n) \approx (I_n, J_n(\lambda)) \quad \text{and} \quad (N_n, I_n) \approx (J_n(0), I_n)$$

are valid since N_n is similar to $J_n(0)$. The pair (24) is (18) with $n = 2k + 1$ and $\varepsilon = 0$; by (20) it is equivalent to $(F_k, G_k) \oplus (F_k^T, G_k^T)$. \square

4 Pairs of matrices, in which the first is symmetric and the second is skew-symmetric

Theorem 4.1. *Over an algebraically closed field \mathbb{F} of characteristic different from 2, every pair (A, B) of matrices of the same size, in which A is symmetric and B is skew-symmetric, is congruent to a direct sum, determined*

uniquely up to permutation of summands, of tridiagonal pairs of three types:

$$\left(\begin{bmatrix} 0 & 1 & & 0 \\ 1 & 0 & 1 & \\ & 1 & 0 & \ddots \\ & & \ddots & \ddots & 1 \\ 0 & & & 1 & 0 \end{bmatrix}_{2k}, \begin{bmatrix} 0 & \lambda & & 0 \\ -\lambda & 0 & \lambda & \\ & -\lambda & 0 & \ddots \\ & & \ddots & \ddots & \lambda \\ 0 & & & -\lambda & 0 \end{bmatrix}_{2k} \right), \quad \begin{matrix} \lambda \in \mathbb{F}, \\ \lambda \neq 0, \end{matrix} \quad (26)$$

in which λ is determined up to replacement by $-\lambda$;

$$\left(\begin{bmatrix} \varepsilon & 0 & & 0 \\ 0 & 0 & 1 & \\ & 1 & 0 & 0 \\ & & 0 & 0 & 1 \\ & & & 1 & 0 & \ddots \\ 0 & & & & \ddots & \ddots \end{bmatrix}_n, \begin{bmatrix} 0 & 1 & & 0 \\ -1 & 0 & 0 & \\ & 0 & 0 & 1 \\ & & -1 & 0 & 0 \\ & & & 0 & 0 & \ddots \\ 0 & & & & \ddots & \ddots \end{bmatrix}_n \right), \quad (27)$$

in which $\varepsilon = 1$ if n is a multiple of 4, and $\varepsilon \in \{0, 1\}$ otherwise; and

$$\left(\begin{bmatrix} 0 & 1 & & 0 \\ 1 & 0 & 0 & \\ & 0 & 0 & 1 \\ & & 1 & 0 & 0 \\ & & & 0 & 0 & \ddots \\ 0 & & & & \ddots & \ddots \end{bmatrix}_{4k}, \begin{bmatrix} 0 & 0 & & 0 \\ 0 & 0 & 1 & \\ & -1 & 0 & 0 \\ & & 0 & 0 & 1 \\ & & & -1 & 0 & \ddots \\ 0 & & & & \ddots & \ddots \end{bmatrix}_{4k} \right). \quad (28)$$

(b) This direct sum is determined uniquely up to permutation of summands by the Kronecker canonical form of (A, B) under equivalence. The Kronecker canonical form of each of the direct summands is given in the following table:

Pair	Kronecker canonical form of the pair
(26)	$(I_k, J_k(\lambda)) \oplus (I_k, J_k(-\lambda))$ with $\lambda \neq 0$
(27) with $\varepsilon = 0$	$(F_k, G_k) \oplus (F_k^T, G_k^T)$ if $n = 2k + 1$ $(J_k(0), I_k) \oplus (J_k(0), I_k)$ if $n = 2k$ (k is odd)
(27) with $\varepsilon = 1$	$(I_n, J_n(0))$ if n is odd $(J_n(0), I_n)$ if n is even
(28)	$(I_{2k}, J_{2k}(0)) \oplus (I_{2k}, J_{2k}(0))$

(29)

Proof. The Kronecker canonical form of (A, B) is a direct sum of pairs of the types:

- (i) $(I_k, J_k(\lambda)) \oplus (I_k, J_k(-\lambda))$, in which $\lambda \neq 0$ if k is odd,
- (ii) $(I_n, J_n(0))$ with odd n ,
- (iii) $(J_k(0), I_k) \oplus (J_k(0), I_k)$ with odd k ,
- (iv) $(J_n(0), I_n)$ with even n ,
- (v) $(F_k, G_k) \oplus (F_k^T, G_k^T)$.

This statement was proved in [24, Section 4] for pairs of complex matrices and goes back to Kronecker's 1874 paper; see the historical remark at the end of Section 4 in [24]. The proof remains valid for matrix pairs over \mathbb{F} (or see [22, Theorem 4]).

In view of Lemma 2.2, it suffices to prove (29).

By Lemma 2.3, (26) is equivalent to

$$(J_k(1), -\lambda J_k(-1)) \oplus (J_k(1), \lambda J_k(-1)),$$

which is equivalent to (i) with $\lambda \neq 0$.

The pair (27) with $n = 2k + 1$ has the form $([\varepsilon] \oplus M_k^+, M_k^- \oplus 0_1)$; by (14) and (15) this pair is equivalent to (v) if $\varepsilon = 0$ or (ii) if $\varepsilon = 1$.

The pair (27) with $n = 2k$ has the form

$$([\varepsilon] \oplus M_{k-1}^+ \oplus 0_1, M_k^-), \tag{30}$$

in which $\varepsilon \in \{0, 1\}$ if k is odd and $\varepsilon = 1$ if k is even. Due to (16) and (17), (30) is equivalent to (iii) if $\varepsilon = 0$ or to (iv) if $\varepsilon = 1$.

The pair (28) has the form $(M_{2k}^+, 0_1 \oplus M_{2k-1}^- \oplus 0_1)$, and by (16) it is equivalent to (i) with $\lambda = 0$ and k replaced by $2k$. \square

5 Pairs of skew-symmetric matrices

Theorem 5.1. *Over an algebraically closed field \mathbb{F} of characteristic different from 2, every pair (A, B) of skew-symmetric matrices of the same size is congruent to a direct sum, determined uniquely up to permutation of summands,*

of tridiagonal pairs of two types:

$$\left(\begin{bmatrix} 0 & 1 & & & 0 \\ -1 & 0 & 0 & & \\ & 0 & 0 & 1 & \\ & & -1 & 0 & 0 \\ & & & 0 & 0 & \ddots \\ 0 & & & & \ddots & \ddots \end{bmatrix}_{2k}, \begin{bmatrix} 0 & \lambda & & & 0 \\ -\lambda & 0 & 1 & & \\ & -1 & 0 & \lambda & \\ & & -\lambda & 0 & 1 \\ & & & -1 & 0 & \ddots \\ 0 & & & & \ddots & \ddots \end{bmatrix}_{2k} \right) \quad (31)$$

and

$$\left(\begin{bmatrix} 0 & 0 & & & 0 \\ 0 & 0 & 1 & & \\ & -1 & 0 & 0 & \\ & & 0 & 0 & 1 \\ & & & -1 & 0 & \ddots \\ 0 & & & & \ddots & \ddots \end{bmatrix}_n, \begin{bmatrix} 0 & 1 & & & 0 \\ -1 & 0 & 0 & & \\ & 0 & 0 & 1 & \\ & & -1 & 0 & 0 \\ & & & 0 & 0 & \ddots \\ 0 & & & & \ddots & \ddots \end{bmatrix}_n \right) \quad (32)$$

in which $k, n \in \mathbb{N}$ and $\lambda \in \mathbb{F}$.

(b) This direct sum is determined uniquely up to permutation of summands by the Kronecker canonical form of (A, B) under equivalence. The Kronecker canonical form of each of the direct summands is given in the following table:

Pair	Kronecker canonical form of the pair
(31)	$(I_k, J_k(\lambda)) \oplus (I_k, J_k(\lambda))$
(32)	$(F_k, G_k) \oplus (F_k^T, G_k^T)$ if $n = 2k + 1$ $(J_k(0), I_k) \oplus (J_k(0), I_k)$ if $n = 2k$

(33)

Proof. The Kronecker canonical form of (A, B) under equivalence is a direct sum of pairs of three types:

$$\begin{aligned} &((I_k, J_k(\lambda)) \oplus (I_k, J_k(\lambda))), \quad ((J_k(0), I_k) \oplus (J_k(0), I_k)), \\ &((F_k, G_k) \oplus (F_k^T, G_k^T)). \end{aligned}$$

This statement was proved in [24, Section 4] for pairs of complex matrices, but the proof remains valid for pairs over \mathbb{F} (or see [22, Theorem 4]). In view of Lemma 2.2, it suffices to prove (33).

The pair (31) has the form $(M_k^-, \lambda M_k^- + (0_1 \oplus M_{k-1}^- \oplus 0_1))$ and by (16) it is equivalent to

$$(I_k, \lambda I_k + J_k(0)) \oplus (I_k, \lambda I_k + J_k(0)) = (I_k, J_k(\lambda)) \oplus (I_k, J_k(\lambda)).$$

The pair (32) with $n = 2k + 1$ has the form $(0_1 \oplus M_k^-, M_k^- \oplus 0_1)$; by (14) it is equivalent to $(F_k, G_k) \oplus (F_k^T, G_k^T)$.

The pair (32) with $n = 2k$ has the form $(0_1 \oplus M_{k-1}^- \oplus 0_1, M_k^-)$; by (16) it is equivalent to $(J_k(0), I_k) \oplus (J_k(0), I_k)$. \square

6 Matrices with respect to congruence

In this section we prove Theorem 1.1.

(a) Each square matrix A can be expressed uniquely as the sum of a symmetric and a skew-symmetric matrix:

$$A = A_{\text{sym}} + A_{\text{sk}}, \quad A_{\text{sym}} := \frac{A + A^T}{2}, \quad A_{\text{sk}} := \frac{A - A^T}{2}.$$

Two matrices A and B are congruent if and only if the corresponding pairs $(A_{\text{sym}}, A_{\text{sk}})$ and $(B_{\text{sym}}, B_{\text{sk}})$ are congruent. Therefore, adding the first and the second matrices in each of the canonical pairs from Theorem 4.1 gives three types of canonical matrices for congruence:

$$\begin{bmatrix} 0 & 1 + \mu & & 0 \\ 1 - \mu & \ddots & \ddots & \\ & \ddots & 0 & 1 + \mu \\ 0 & & 1 - \mu & 0 \end{bmatrix}_{2k}, \quad \begin{array}{l} \mu \neq 0, \\ \mu \text{ is determined up} \\ \text{to replacement by } -\mu; \end{array} \quad (34)$$

(3); and (4). We can assume that $\mu \neq -1$ because the congruence transformation

$$X \mapsto S^T X S, \quad S := \begin{bmatrix} 0 & & 1 \\ & \ddots & \\ 1 & & 0 \end{bmatrix}, \quad (35)$$

maps (34) with $\mu = -1$ into (34) with $\mu = 1$. If we multiply all the odd columns and rows of (34) by $(1 + \mu)^{-1}$ (this is a transformation of congruence), we obtain (2) with

$$\lambda = \frac{1 - \mu}{1 + \mu}. \quad (36)$$

The parameter μ is determined up to replacement by $-\mu$, so each $\lambda \neq 0$ is determined up to replacement by λ^{-1} , whereas $\lambda = 0$ is determined uniquely since it corresponds to $\mu = 1$ and we assume that $\mu \neq -1$. We have $\lambda \neq \pm 1$ because $\mu \neq 0$ and $-1 + \mu \neq 1 + \mu$. The parameter λ is an arbitrary element of \mathbb{F} except for ± 1 since substituting $\mu = (1 - \lambda)/(1 + \lambda)$ into (36) gives the identity.

(b) Let A be the matrix (2). By Lemma 2.3, the pair (A^T, A) is equivalent to

$$\left(\left[\begin{array}{ccc|ccc} \lambda & 1 & & & & \\ & \lambda & \ddots & & & \\ & & \ddots & 1 & & \\ & & & \lambda & & \\ \hline & & & 1 & \lambda & \\ & & & & 1 & \ddots \\ 0 & & & & & \ddots & \lambda \\ & & & & & & 1 \end{array} \right], \left[\begin{array}{ccc|ccc} 1 & \lambda & & & & \\ & 1 & \ddots & & & \\ & & \ddots & \lambda & & \\ & & & 1 & & \\ \hline & & & \lambda & 1 & \\ & & & & \lambda & \ddots \\ 0 & & & & & \ddots & 1 \\ & & & & & & \lambda \end{array} \right] \right), \quad (37)$$

which is equivalent to $(J_k(\lambda), I_k) \oplus (I_k, J_k(\lambda))$ since $\lambda \neq \pm 1$. This verifies the assertion about the matrix (2) in table (5).

The remaining assertions about the matrices (3) and (4) in table (5) follow from the corresponding assertions about the matrices (27) and (28) in table (29): the matrices (3) and (4) have the form $A = B + C$ in which (B, C) is (27) or (28), and so $(A^T, A) = (B - C, B + C)$. For example, if A is (3) with $\varepsilon = 1$, then by (29)

$$(B, C) \approx \begin{cases} (I_n, J_n(0)) & \text{if } n \text{ is odd,} \\ (J_n(0), I_n) & \text{if } n \text{ is even,} \end{cases}$$

and we have

$$(A^T, A) \approx \begin{cases} (I_n - J_n(0), I_n + J_n(0)) \approx (I_n, J_n(1)) & \text{if } n \text{ is odd,} \\ (J_n(0) - I_n, J_n(0) + I_n) \approx (I_n, J_n(-1)) & \text{if } n \text{ is even.} \end{cases}$$

The proof of Theorem 1.1 is complete.

7 Matrices with respect to *-congruence

In this section we prove Theorem 1.2.

Let \mathbb{F} be an algebraically closed field with nonidentity involution represented in the form (10). A canonical form of a square matrix A over \mathbb{F} for *-congruence was given in [22] and was improved in [7] (a direct proof that the matrices in [7] are canonical is given in [8, 9]): A is *-congruent to a direct sum, determined uniquely up to permutation of summands, of matrices of three types:

$$\begin{bmatrix} 0 & I_k \\ J_k(\lambda) & 0 \end{bmatrix} \quad (\lambda \neq 0, |\lambda| \neq 1), \quad \mu \begin{bmatrix} 0 & & & 1 \\ & \ddots & & i \\ & & \ddots & \\ 1 & i & & 0 \end{bmatrix} \quad (|\mu| = 1), \quad J_n(0), \quad (38)$$

in which λ is determined up to replacement by $\bar{\lambda}^{-1}$. It follows from the proof of Theorem 3 in [22] that instead of (38) one can take any set of matrices

$$P_{2k}(\lambda), \quad \mu Q_n, \quad J_n(0)$$

(with the same conditions on λ and μ) such that

$$(P_{2k}(\lambda)^*, P_{2k}(\lambda)) \approx (J_k(\bar{\lambda}), I_k) \oplus (I_k, J_k(\lambda)) \quad (39)$$

and

$$(Q_n^*, Q_n) \approx (I_n, J_n(\nu_n)), \quad (40)$$

in which ν_1, ν_2, \dots are any elements of \mathbb{F} with modulus one.

Proof of Theorem 1.2. Let $P_{2k}(\lambda)$ be the matrix (6) with $\lambda \neq 0$ and let Q_n be the matrix (7) with $\mu = 1$. Since the matrix (6) with $\lambda = 0$ is $J_n(0)$, it suffices to prove that (39) and (40) are fulfilled.

By Lemma 2.3, $(P_{2k}(\lambda)^*, P_{2k}(\lambda))$ is equivalent to the pair (37) with $\bar{\lambda}$ instead of λ in the first matrix. This proves (39) since $|\lambda| \neq 1$.

The matrix Q_n is (3) with $\varepsilon = 1$. Due to (5),

$$(Q_n^*, Q_n) = (Q_n^T, Q_n) \approx (I_n, J_n((-1)^{n+1}));$$

this ensures (40) with $\nu_n := (-1)^{n+1}$.

The assertion about the matrix (6) with $\lambda = 0$ in table (8) follows from the equivalence

$$(J_n(0)^T, J_n(0)) \approx \begin{cases} (F_k, G_k) \oplus (F_k^T, G_k^T) & \text{if } n = 2k + 1, \\ (J_k(0), I_k) \oplus (I_k, J_k(0)) & \text{if } n = 2k, \end{cases}$$

which was established in the proof of Theorem 3 in [22]. \square

8 Pairs of Hermitian matrices

Theorem 8.1. (a) *Over an algebraically closed field \mathbb{F} with nonidentity involution represented in the form (10), every pair (A, B) of Hermitian matrices of the same size is *-congruent to a direct sum, determined uniquely up to permutation of summands, of tridiagonal pairs of two types:*

$$\left(\begin{bmatrix} 0 & 1 & & & 0 \\ 1 & 0 & 1 & & \\ & 1 & 0 & \ddots & \\ & & \ddots & \ddots & 1 \\ 0 & & & 1 & 0 \end{bmatrix}_n, \begin{bmatrix} 0 & \mu & & & 0 \\ \bar{\mu} & 0 & \mu & & \\ & \bar{\mu} & 0 & \ddots & \\ & & \ddots & \ddots & \mu \\ 0 & & & \bar{\mu} & 0 \end{bmatrix}_n \right), \quad (41)$$

in which $\mu \in \mathbb{F} \setminus \mathbb{P}$ if n is even, $\mu = \pm i$ if n is odd, and μ is determined up to replacement by $\bar{\mu}$; and

$$\left(\begin{bmatrix} a & b & & & 0 \\ b & 0 & a & & \\ & a & 0 & b & \\ & & b & 0 & a \\ & & & a & 0 & \ddots \\ 0 & & & & \ddots & \ddots \end{bmatrix}_n, \begin{bmatrix} b & -a & & & 0 \\ -a & 0 & b & & \\ & b & 0 & -a & \\ & & -a & 0 & b \\ & & & b & 0 & \ddots \\ 0 & & & & \ddots & \ddots \end{bmatrix}_n \right), \quad (42)$$

in which $a, b \in \mathbb{P}$ and $a^2 + b^2 = 1$.

(b) *The Kronecker canonical form of (A, B) under equivalence determines this direct sum uniquely up to permutation of summands and multiplication by -1 any direct summand of type (42). The Kronecker canonical form of*

each of the direct summands is given in the following table:

Pair	Kronecker canonical form of the pair
(41)	$(F_k, G_k) \oplus (F_k^T, G_k^T)$ if $n = 2k + 1$ $(I_k, J_k(\mu)) \oplus (I_k, J_k(\bar{\mu}))$ if $n = 2k$
(42)	$(I_n, J_n(b/a))$ if n is odd and $a \neq 0$ $(I_n, J_n(-a/b))$ if n is even and $b \neq 0$ $(J_n(0), I_n)$ otherwise

(43)

Proof. (a) Each square matrix A over \mathbb{F} has a *Cartesian decomposition*

$$A = B + iC, \quad B := \frac{A + A^*}{2}, \quad C := \frac{i(A^* - A)}{2},$$

in which both B and C are Hermitian. Two square matrices A and A' are \ast -congruent if and only if the corresponding pairs (B, C) and (B', C') are \ast -congruent. Therefore, if we apply the Cartesian decomposition to the canonical matrices for \ast -congruence from Theorem 1.2, we obtain canonical pairs of Hermitian matrices for \ast -congruence. To simplify these canonical pairs, we multiply (6) by 2 (this is a transformation of \ast -congruence), and using (10) take μ in (7) to have the form $a + bi$ with $a, b \in \mathbb{P}$. Thus, every pair (A, B) of Hermitian matrices of the same size is \ast -congruent to a direct sum, determined uniquely up to permutation of summands, of pairs of two types:

$$\left(\begin{bmatrix} 0 & 1 + \bar{\lambda} & & 0 \\ 1 + \lambda & 0 & \ddots & \\ & \ddots & \ddots & 1 + \bar{\lambda} \\ 0 & & 1 + \lambda & 0 \end{bmatrix}_n, i \begin{bmatrix} 0 & \bar{\lambda} - 1 & & 0 \\ 1 - \lambda & 0 & \ddots & \\ & \ddots & \ddots & \bar{\lambda} - 1 \\ 0 & & 1 - \lambda & 0 \end{bmatrix}_n \right), \quad (44)$$

in which $\lambda \in \mathbb{F}$, $|\lambda| \neq 1$, each nonzero λ is determined up to replacement by $\bar{\lambda}^{-1}$, and $\lambda = 0$ if n is odd; and

$$\left(\begin{bmatrix} a & bi & & 0 \\ -bi & 0 & a & \\ & a & 0 & bi \\ & & -bi & 0 & a \\ & & & a & 0 & \ddots \\ 0 & & & & \ddots & \ddots \end{bmatrix}_n, \begin{bmatrix} b & -ai & & 0 \\ ai & 0 & b & \\ & b & 0 & -ai \\ & & ai & 0 & b \\ & & & b & 0 & \ddots \\ 0 & & & & \ddots & \ddots \end{bmatrix}_n \right), \quad (45)$$

in which $a^2 + b^2 = 1$.

Let us prove that the pairs (44) and (45) are \ast -congruent to the pairs (41) and (42).

We obtain (42) if we apply the \ast -congruence transformation $X \mapsto S^\ast XS$ with

$$S := \text{diag}(1, -i, -i, -1, -1, i, i, 1, 1, -i, -i, -1, -1, \dots)$$

to the matrices of (45).

The pair (44) with $\lambda = 0$ is the pair (41) with $\mu = -i$, which is \ast -congruent to (41) with $\mu = i$ via the transformation (35).

It remains to consider (44) with $\lambda \neq 0$. Then n is even. Applying to the matrices of (44) the \ast -congruence transformation $X \mapsto S^\ast XS$ with

$$S := \text{diag} \left(1, \frac{1}{1 + \bar{\lambda}}, \frac{1 + \lambda}{1 + \bar{\lambda}}, \frac{1 + \lambda}{(1 + \bar{\lambda})^2}, \frac{(1 + \lambda)^2}{(1 + \bar{\lambda})^2}, \frac{(1 + \lambda)^2}{(1 + \bar{\lambda})^3}, \dots \right),$$

(the denominator is nonzero since $|\lambda| \neq 1$), we obtain (41) with

$$\mu := \frac{\bar{\lambda} - 1}{\bar{\lambda} + 1} i. \quad (46)$$

Since λ is nonzero and is determined up to replacement by $\bar{\lambda}^{-1}$, we have that $\mu \neq -i$ and μ is determined up to replacement by

$$\frac{\lambda^{-1} - 1}{\lambda^{-1} + 1} i = \frac{1 - \lambda}{1 + \lambda} i = \bar{\mu}.$$

Every $\mu \in \mathbb{F}$ except for i can be represented in the form (46) with $\lambda = (i - \bar{\mu})/(i + \bar{\mu})$. We do not impose the condition $\mu \neq \pm i$ in (41) because (41) with $\mu = \pm i$ is \ast -congruent to (44) with $\lambda = 0$.

Let us prove that the condition $|\lambda| \neq 1$ is equivalent to the condition $\mu \notin \mathbb{P}$. If $|\lambda| = 1$ and $\lambda = a + bi \neq -1$ with $a, b \in \mathbb{P}$, then

$$\mu = \frac{(\bar{\lambda} - 1)(\lambda + 1)}{(\bar{\lambda} + 1)(\lambda + 1)} i = \frac{\bar{\lambda}\lambda - \lambda + \bar{\lambda} - 1}{\bar{\lambda}\lambda + \lambda + \bar{\lambda} + 1} i = \frac{-bi}{1 + a} i \in \mathbb{P}. \quad (47)$$

Each $\mu \in \mathbb{P}$ can be represented in the form (47) as follows: $\mu = b/(1 + a)$, in which

$$a := \frac{1 - \mu^2}{1 + \mu^2} \quad \text{and} \quad b := \frac{2\mu}{1 + \mu^2} \quad (\text{then } a^2 + b^2 = 1).$$

(b) Lemma 2.3 ensures the assertion about the pair (41) in table (43).

The pair (42) has the form $(aX+bY, bX-aY)$, in which (X, Y) is (19) with $\lambda = 0$. By (20), $(X, Y) \approx (I_n, J_n(0))$ if n is odd, and $(X, Y) \approx (J_n(0), I_n)$ if n is even. Therefore,

$$\text{Pair (42)} \approx \begin{cases} (aI_n + bJ_n(0), bI_n - aJ_n(0)) & \text{if } n \text{ is odd,} \\ (aJ_n(0) + bI_n, bJ_n(0) - aI_n) & \text{if } n \text{ is even.} \end{cases}$$

This validates the assertion about the pair (42) in table (43). \square

Remark 8.1. The pair (42) with two dependent parameters, which was obtained from the Cartesian decomposition of (7), can be replaced by 0- and 1-parameter matrices as follows. The matrices (7) have the form μA , in which $\mu = a + bi$, $a, b \in \mathbb{P}$, and $a^2 + b^2 = 1$. If $\mu \neq \pm i$, then $a \neq 0$ and μA is *congruent to $|a|^{-1}\mu A = \pm(1 + ci)A$ with $c \in \mathbb{P}$. Now apply the Cartesian decomposition to $\pm iA$ and $\pm(1 + ci)A$ with $c \in \mathbb{P}$.

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